# String Stability of Interconnected Systems

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*Abstract*— In this paper we introduce the notion of string stability of a countably infinite interconnection of a class of nonlinear systems. Intuitively, string stability implies uniform boundedness of all the states of the interconnected system for all time if the initial states of the interconnected system are uniformly bounded. It is well known that the input–output gain of all the subsystems less than unity guarantees that the interconnected system is input–output stable. We derive sufficient ("weak coupling") conditions which guarantee the asymptotic string stability of a class of interconnected systems. Under the same "weak coupling" conditions, string-stable interconnected systems remain string stable in the presence of small structural/singular perturbations. In the presence of parameter mismatch, these "weak coupling" conditions ensure that the states of all the subsystems are all uniformly bounded when a gradient-based parameter adaptation law is used and that the states of all the systems go to zero asymptotically.

## I. INTRODUCTION

**E**ARLIER research on interconnected systems focused on vehicle-following applications [17], [14], [8], [23], [11], control of distributed systems (e.g., regulation of seismic cables, vibration control in beams, etc.) [7], [19], signal processing [4], and power systems [5]. Loosely speaking, string stability of an interconnected system implies uniform boundedness of the state of all the systems. For example, in automated vehicle-following applications, tracking (spacing) errors should not amplify downstream from vehicle to vehicle for safety. Similarly, deflection at any point in a beam or a rod should remain bounded at all times. Spatial discretization and control of such distributed systems have a relevance to the problem of string stability for interconnected systems. Although a precise definition of string stability was not coined, Kuo and Melzer [17] and Levine and Athans [14] were seeking optimal control solutions to the automated vehicle-following problem. Chu defined string stability in the context of vehicle following [11]. In [4], Chang introduces a stronger version of stability for interconnected systems, namely, "$\gamma$-stability" for infinite interconnection of linear digital processors. Intuitively, "$\gamma$-stability" ensures that the state of all the systems decays to zero exponentially in time and system index. In this paper, we generalize the concept of string stability to a class of interconnected systems and seek sufficient conditions to guarantee their string stability. We also examine their robustness to structural and singular perturbations.

This paper is organized as follows: In Section I, we define string stability and asymptotic string stability, we present "weak coupling" conditions that guarantee string stability for a class of interconnected systems, and we demonstrate that exponential string stability is preserved under small structural perturbations. In Section II, we prove that every exponentially string-stable interconnected system is string stable in the presence of small singular perturbations. In Section III, we discuss direct adaptive control of such interconnected systems. In Section IV, we provide an example of longitudinal controller design for vehicle-following systems.

## II. STRING STABILITY

We use the following notations: $\|f_i(\cdot)\|_\infty$, or simply $\|f_i\|_\infty$ denotes $\sup_{t \geq 0} |f_i(t)|$, and $\|f_i(0)\|_\infty$ denotes $\sup_i |f_i(0)|$. For all $p < \infty$, $\|f_i(\cdot)\|_p$ or $\|f_i\|_p$ denotes $\left(\int_0^\infty |f_i(t)|^p \, dt\right)^{\frac{1}{p}}$ and $\|f_i(0)\|_p$ denotes $\left(\sum_1^\infty |f_i(0)|^p\right)^{\frac{1}{p}}$.

Consider the following interconnected system:

$$\dot{x}_i = f(x_i, x_{i-1}, \cdots, x_{i-r+1}) \tag{1}$$

where $i \in \mathcal{N}$, $x_{i-j} \equiv 0 \; \forall i \leq j$, $x \in \mathcal{R}^n$, $f : \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{r \text{ times}} \to \mathcal{R}^n$ and $f(0, \cdots, 0) = 0$.

*Definition 1:* The origin $x_i = 0$, $i \in \mathcal{N}$ of (1) is string stable, if given any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x_i(0)\|_\infty < \delta \Rightarrow \sup_i \|x_i(\cdot)\|_\infty < \epsilon$.

*Definition 2:* The origin $x_i = 0$, $i \in \mathcal{N}$ of (1) is asymptotically (exponentially) string stable if it is string stable and $x_i(t) \to 0$ asymptotically (exponentially) for all $i \in \mathcal{N}$.

A more general definition of string stability is the following one.

*Definition 3 ($l_p$ String Stability):* The origin $x_i = 0$, $i \in \mathcal{N}$ of (1) is $l_p$ string stable if for all $\epsilon > 0$, there exists a $\delta$ such that

$$\|x_i(0)\|_p < \delta \Leftrightarrow \sup_t \left(\sum_1^\infty |x_i(t)|^p\right)^{\frac{1}{p}} < \epsilon.$$

Definition 1 of string stability can be restated as $l_\infty$-string stability of Definition 3. Henceforth, we will deal with string stability according to Definition 1. The following theorem proves, under some "weak coupling" conditions, that any countably infinite interconnection of exponentially stable nonlinear systems is string stable. Clearly, a string of uncoupled exponentially stable systems is exponentially string stable. Intuitively, any interconnection of exponentially stable systems is string stable, if the interconnections are sufficiently weak. The following lemmas will be useful in proving the theorems in this paper.

*Lemma 1:* Let $r$ be a constant positive integer. Define $P_r(z) = z^r - \sum_1^r \beta_j z^{r-j}$, $\beta_j > 0$. If $\sum_1^r \beta_j < 1$, the $r$th degree polynomial $P_r(z)$ has all its roots inside the unit circle.

*Proof:* Let $z_0$ be such that $P_r(z_0) = 0$ and $|z_0| > 1$. Then

$$1 = \sum_1^r \beta_j z_0^{-j} \leq \sum_1^r \beta_j < 1$$

which is a contradiction. This proves the lemma.

*Lemma 2:* Let $V_i(t) \geq 0 \ \forall t \geq 0$, $i \in \mathcal{N}$ and if

$$\dot{V}_i \leq -\beta_0 V_i + \sum_1^\infty \beta_j V_{i-j}$$

with $\beta_0 > 0$ and $\beta_j \geq 0$, $j = 1, 2, \cdots$ and $\beta_0 > \sum_1^\infty \beta_j$. For all $j \leq 0$, $V_j$ should be read as zero. Then, given any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\|V_i(0)\|_\infty < \delta \Rightarrow \sup_i \|V_i\|_\infty < \epsilon.$$

*Proof:* Let $M = \dfrac{\beta_0}{\beta_0 - \sum_1^\infty \beta_j} > 1$. It suffices to show that $\|V_i\|_\infty \leq M\|V_i(0)\|_\infty$. We prove this by induction. From the inequality, it follows that:

$$\|V_i\|_\infty \leq V_i(0) + \sum_1^\infty \frac{\beta_j}{\beta_0}\|V_{i-j}\|_\infty.$$

For $i = 1$, $\|V_i\|_\infty \leq V_i(0)$ and the induction hypothesis is valid. Assuming that the hypothesis is valid through the integer $i$

$$\|V_{i+1}\|_\infty \leq V_{i+1}(0) + \sum_1^\infty \frac{\beta_j}{\beta_0} M\|V_i(0)\|_\infty$$
$$\leq \left(1 + M\sum_1^\infty \frac{\beta_j}{\beta_0}\right)\|V_i(0)\|_\infty$$
$$= M\|V_i(0)\|_\infty.$$

This proves that the induction hypothesis is valid for all $i$. Therefore, $\sup_i \|V_i\|_\infty \leq M\|V_i(0)\|_\infty$.

*Theorem 1 (Weak Coupling Theorem for String Stability):* If the following conditions are satisfied:

- $f$ is globally Lipschitz in its arguments, i.e.,

$$|f(y_1, \cdots, y_r) - f(z_1, \cdots, z_r)|$$
$$\leq l_1|y_1 - z_1| + \cdots + l_r|y_r - z_r|. \quad (2)$$

- The origin of $\dot{x} = f(x, 0, \cdots, 0)$ is globally exponentially stable.

Then for sufficiently small $l_i$, $i = 2, \cdots, r$, the interconnected system is globally exponentially string stable.

*Proof:* Since the origin of $\dot{x} = f(x, 0, \cdots, 0)$ is exponentially stable, by the converse Lyapunov theorem, there exists a Lyapunov function $V(x)$ and four positive constants $\alpha_l$, $\alpha_h$, $\alpha_1$, $\alpha_3$ such that

$$\alpha_l\|x\|^2 \leq V(x) \leq \alpha_h\|x\|^2 \quad (3)$$
$$\frac{\partial V}{\partial x}f(x, 0, \cdots, 0) \leq -\alpha_1\|x\|^2 \quad (4)$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_3\|x\|. \quad (5)$$

For the sake of convenience, we denote $V(x_i)$ by $V_i$. Then

$$\dot{V}_i = \frac{\partial V_i}{\partial x_i}f(x_i, x_{i-1}, \cdots, x_{i-r+1})$$
$$= \frac{\partial V_i}{\partial x_i}f(x_i, 0, \cdots, 0)$$
$$+ \frac{\partial V_i}{\partial x_i}[f(x_i, x_{i-1}, \cdots, x_{i-r+1}) - f(x_i, 0, \cdots, 0)]$$
$$\leq -\alpha_1\|x_i\|^2 + \alpha_3\|x_i\|\left(\sum_{j=2}^r l_j\|x_{i-j+1}\|\right).$$

Using the inequality that $xy \leq \dfrac{x^2+y^2}{2}$, the above equation results in

$$\dot{V}_i \leq -\frac{\left(\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j\right)}{\alpha_h}V_i + \frac{\alpha_3}{2\alpha_l}\sum_{j=2}^r l_j V_{i-j+1}. \quad (6)$$

If $\sum_{j=2}^r l_j$ is sufficiently small such that $\sum_{j=2}^r l_j < \dfrac{2\alpha_l\alpha_1}{\alpha_3(\alpha_l + \alpha_h)}$, then $\dfrac{\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j}{\alpha_h} > \dfrac{\alpha_3}{2\alpha_l}\sum_{j=2}^r l_j > 0$. Consequently, string stability follows from Lemmas 1 and 2.

Let $d > 1$. Define $V(d^{-1}, t) = \sum_{j=1}^\infty V_i(t)d^{-i}$. Clearly, $V(d^{-1}, t)$ is defined whenever the weak coupling conditions are satisfied and whenever $\|x_i(0)\|_\infty$ exists

$$\dot{V} = \sum_{j=1}^\infty \dot{V}_i(t)d^{-i} \leq -Vd^{-(r-1)}P_r(d)\frac{\alpha_1 - \frac{\alpha_3}{2}\sum_{j=2}^r l_j}{\alpha_h}.$$

Here $P_r(z) = z^r - \sum_1^r \beta_j z^{r-j}$ where $\beta_j = \dfrac{\alpha_3}{2\alpha_l}\dfrac{\alpha_h}{\alpha_1 - \frac{\alpha_3}{2}\sum l_j}l_j$. Clearly, $P_r(d) > 0$ whenever $d > 1 > \rho(P_r(z))$, the spectral radius of the polynomial $P_r(z)$. $V \to 0$ exponentially and hence, $V_i(t)$, $x_i(t) \to 0$ exponentially.

The above theorem can easily be generalized to nonautonomous interconnected systems. Consider the following nonautonomous interconnection:

$$\dot{x}_i = f(x_i, x_{i-1}, \cdots, x_{i-r+1}, t)$$

where $i \in \mathcal{N}$, $x_{i-j} \equiv 0 \ \forall i \leq j$, $x \in \mathcal{R}^n$, $f: \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{r \text{ times}} \times \mathcal{R} \to \mathcal{R}^n$ and $f(0, \cdots, 0) = 0$.

*Remark:* If the following conditions are satisfied:
- $f$ is globally Lipschitz in its arguments, i.e.,

$$|f(y_1, \cdots, y_r, t) - f(z_1, \cdots, z_r, t)|$$
$$\leq l_1|y_1 - z_1| + \cdots + l_r|y_r - z_r|.$$

- The origin of $\dot{x} = f(x, 0, \cdots, 0, t)$ is globally exponentially stable, i.e., there exists a Lyapunov function, $V(x)$ such that

$$\alpha_l\|x\|^2 \leq V(x, t) \leq \alpha_h\|x\|^2$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}f(x, \cdots, 0, t) \leq -\alpha_3\|x\|^2$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4\|x\|.$$

Then for sufficiently small $l_i$, $i = 2, \cdots, r$, the interconnected system is globally exponentially string stable.

*Proof:* Let $V(x_i, t) = V_i$. Clearly, $\dot{V}_i \leq -\alpha_3\|x_i\|^2 + \alpha_4\|x_i\|\sum_1^r l_j\|x_{i-j}\|$. By the same arguments used in Theorem 1, the desired conclusion follows.

Another simple class of interconnected systems that arise in the context of vehicle-following systems is given by

$$\dot{x}_i = f(x_i, x_{i-1}, \dot{x}_{i-1}) \qquad (7)$$

where $i \in \mathcal{N}$, $x_{i-j} \equiv 0 \,\forall\, i \leq j$, $x \in \mathcal{R}^n$, $f: \mathcal{R}^n \times \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^n$ and $f(0, 0, 0) = 0$. If the following conditions are satisfied:

* $f$ is globally Lipschitz in its arguments, i.e.,

$$|f(y_1, y_2, y_3) - f(z_1, z_2, z_3)| \leq$$
$$l_1|y_1 - z_1| + \cdots + l_2|y_2 - z_2| + d_1|y_3 - z_3|$$

* The origin of $\dot{x} = f(x, 0, 0)$ is globally exponentially stable, i.e., there exists a Lyapunov function, $V(x)$ such that

$$\alpha_l\|x\|^2 \leq V(x) \leq \alpha_h\|x\|^2$$
$$\frac{\partial V}{\partial x}f(x, 0, 0) \leq -\alpha_3\|x\|^2$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_4\|x\|.$$

Then for sufficiently small $l_2 + d_1$, the interconnected system is globally exponentially string stable.

*Proof:* From the Lipschitz property

$$\|\dot{x}_i\| \leq l_1\|x_i\| + (l_2 + d_1 l_1)$$
$$\cdot [\|x_{i-1}\| + d_1\|x_{i-2}\| + \cdots + d_1^{i-2}\|x_1\|].$$

Let $V(x_i) = V_i$. Then

$$\dot{V}_i \leq -\frac{\alpha_3}{2\alpha_h}V_i + \frac{\alpha_4}{2\alpha_l}(l_2 + d_1 l_1)[(1 + d_1 + \cdots + d_1^{i-2})V_i$$
$$+ (V_{i-1} + d_1 V_{i-2} + \cdots + d_1^{i-2}V_1)].$$

By the above lemma, string stability follows for sufficiently small $l_1 + l_2 + d_1$. Define $V(t) = \sum_1^\infty V_i d^{-1}$ for any $d > 1$. Then, $\dot{V} \leq -KV$ where $K > 0$. Consequently, exponential stability is guaranteed.

From the definition of string stability, it is clear that the string stability of an interconnected system guarantees the stability of every subsystem. Under some stronger coupling condition, $\alpha_1 > \frac{\alpha_3}{2}\sum_{j=2}^r l_j$, any finite interconnections of one is asymptotically string stable. In the vehicle-following applications, although the number of vehicles in every platoon (electronically interconnected system of vehicles) will be finite, it is necessary that the stability of the platoon be independent of the size of the platoon to prevent the saturation of the input actuators. Another interesting feature about the string stability of an interconnection of exponentially stable systems is that it is preserved under small structural perturbations. Consider

$$\dot{x}_i = f(x_i, \cdots, x_{i-r+1}) + \epsilon f_p(x_i, \cdots, x_{i-r+1}).$$

Assume $f_p(0, \cdots, 0) = 0$ and $\|f_p(p_1, \cdots, p_r) - f_p(q_1, \cdots, q_r)\| \leq \sum_{j=1}^r l_{fj}\|p_j - q_j\|$. From Theorem 1, the interconnection of the perturbed systems is string stable if

$$(\alpha_1 - \alpha_3\epsilon l_{f1}) - \sum_{j=2}^r \frac{\alpha_3(l_j + \epsilon l_{fj})}{2} > \frac{\alpha_h}{\alpha_l}\sum_{j=2}^r \frac{\alpha_3(l_j + \epsilon l_{fj})}{2}.$$

This condition is satisfied when

$$\epsilon < \frac{\alpha_1 - \frac{\alpha_3(\alpha_l + \alpha_h)}{2\alpha_l}\sum_{j=2}^r l_j}{\alpha_3 l_{f1} + \frac{\alpha_3(\alpha_l + \alpha_h)}{2\alpha_l}\sum_{j=2}^r l_{fj}}.$$

This concludes the proof that string stability is robust to small structural perturbations.

*Remark (Weak Coupling for $l_2$ string stability):* Consider the following interconnected system in which every subsystem is connected only to its neighboring subsystems:

$$\dot{x}_i = f(x_{i-1}, x_i, x_{i+1}) \qquad i \in \mathcal{N}.$$

As before, $x_i \equiv 0 \,\forall\, i \leq 0$. If the following conditions are satisfied:

* $f$ is globally Lipschitz in its arguments, i.e.,

$$|f(x_1, x_2, x_3) - f(y_1, y_2, y_3)|$$
$$\leq l_1|x_1 - y_1| + l_2|x_2 - y_2| + l_3|x_3 - y_3|.$$

* The origin of $\dot{x} = f(0, x, 0)$ is globally exponentially stable.

Then for sufficiently small $l_1 + l_3$, the interconnected system is globally exponentially $l_2$ string stable.

*Proof:* Since the origin of $\dot{x} = f(0, x, 0)$ is exponentially stable, by the converse Lyapunov theorem, there exists a Lyapunov function $V(x)$ such that

$$\alpha_l\|x\|^2 \leq V(x) \leq \alpha_h\|x\|^2$$
$$\frac{\partial V}{\partial x}f(0, x, 0) \leq -\alpha_1\|x\|^2$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_3\|x\|.$$

For the sake of convenience we denote $V(x_i)$ by $V_i$. As in Theorem 1, we obtain

$$\dot{V}_i \leq -\frac{(\alpha_1 - \frac{\alpha_3}{2}(l_1 + l_3))}{\alpha_h}V_i + \frac{\alpha_3}{2\alpha_l}(l_1 V_{i-1} + l_3 V_{i+1}).$$

Define $V(d, t) = \sum_1^\infty d^{-i}V_i(t)$ for $d$ sufficiently close to and greater than unity. Note that $V(d, 0)$ is defined if $\|x_i(0)\|_2$ is defined. Differentiating $V$, we obtain

$$\dot{V}(d, t) = \sum_1^\infty d^{-i}\dot{V}_i$$
$$\leq -\left(\frac{\alpha_1 - \frac{\alpha_3}{2}(l_1 + l_3)}{\alpha_h} - \frac{\alpha_3}{2\alpha_l}(l_1 d^{-1} + l_3 d)\right)V.$$

Define $P_2(d) = \frac{(\alpha_1 - \frac{\alpha_3}{2}(l_1 + l_3))}{\alpha_h} - \frac{\alpha_3}{2\alpha_l}(l_1 d^{-1} + l_3 d)$. If $l_1 + l_3 < \frac{2\alpha_1\alpha_l}{\alpha_3(\alpha_l + \alpha_h)}$, then $P_2(1) > 0$ and $P_2(d_0) < 0$ for all $d_0 \geq \frac{2\alpha_1\alpha_l}{\alpha_h\alpha_3 l_3}$. By the intermediate value theorem, there exists a $d^* > 1$ such that $P_2(d^*) = 0$ and $P_2(d) > 0$ for all

$1 < d < d^*$. As a result

$$\dot{V}(d, t) \leq -P_2(d)V(d, t) \leq 0 \qquad 1 < d < d^*$$

which implies that

$$V(d, t) \leq V(d, 0)e^{-P_2(d)t} \leq V(d, 0).$$

This guarantees exponential $l_2$-string stability of the interconnected system. $d^*$ is a performance measure associated with this interconnected system. For a general interconnected system, only $l_2$-string stability can be guaranteed.

It is desirable that the string-stability property be preserved in the presence of parasitic actuator dynamics. In the next section, we present the conditions which guarantee string stability of the origin of the interconnected system in the presence of such parasitic actuator dynamics. From here on, we consider "look-ahead or lower-triangular systems" only, and therefore the results would apply for $l_\infty$-string stability. The results for lower-triangular systems that follow can easily be extended to $l_2$-string stability of general interconnected systems.

## III. STRING STABILITY OF SINGULARLY PERTURBED INTERCONNECTED SYSTEMS

Before proceeding to study the string stability of the interconnected system, we present a result on the stability of a singularly perturbed system from [13].

*Theorem 2 (Robustness of Exponentially Stable Nonlinear Systems to Singular Perturbations):* Consider the autonomous singularly perturbed system

$$\dot{x} = f_1(x, z) \tag{8}$$

$$\epsilon \dot{z} = g_1(x, z) \tag{9}$$

where $x \in \mathcal{R}^n$, $z \in \mathcal{R}^m$ and assume that the origin is an isolated equilibrium point and the functions $f_1$ and $_1$ are locally Lipschitz in an open connected set that contains the origin. Let $z = h_1(x)$ be an isolated root of $0 = g(x, z)$, such that $h_1(0) = 0$. Let $y = z - h_1(x)$. If the following conditions are satisfied:

- The reduced system is exponentially stable, i.e., there exists positive constants $\alpha_l$, $\alpha_h$, $\alpha_1$, $\alpha_3$ and a Lyapunov function $V(x)$ such that

$$\alpha_1\|x\|^2 \leq V(x) \leq \alpha_h\|x\|^2$$
$$\frac{\partial V}{\partial x}f_1(x, h_1(x)) \leq \alpha_1\|x\|^2$$
$$\left\|\frac{\partial V}{\partial x}\right\| \leq \alpha_3\|x\|.$$

- The boundary layer system is exponentially stable, uniformly for frozen $x$, i.e., there exists positive constants $\beta_l$, $\beta_h$, $\alpha_2$, $\alpha_1$, and a Lyapunov function $W(x, y)$ such that

$$\beta_l\|y\|^2 \leq W(x, y) \leq \beta_h\|y\|^2$$
$$\frac{\partial W}{\partial y}g(x, y + h_1(x)) \leq -\alpha_2\|y\|^2$$
$$\left\|\frac{\partial W}{\partial(x, y)}\right\| \leq \alpha_4\|x \quad \dot{y}\|.$$

- There exist positive constants, $\beta_2$ and $\gamma$ such that

$$\left[\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y}\frac{\partial h_1}{\partial x}\right]f_1(x, y + h_1(x)) \leq \beta_2\|x\| \|y\| + \gamma\|y\|^2.$$

Let $\epsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma + \beta_1\beta_2}$. Then the origin of the singularly perturbed system is exponentially stable for $0 < \epsilon < \epsilon^*$.

*Proof:* See Theorem 2.1 and [13, Corollary 2.2].

Intuitively, the origin of the perturbed interconnected system will be string stable if the origin of every perturbed subsystem is stable and the origin of the "reduced" interconnected system is string stable. This observation leads us to the following theorem.

Consider the following perturbed interconnected system:

$$\dot{x}_i = f(x_i, z_i, x_{i-1}, \cdots, x_{i-r+1}) \tag{10}$$

$$\epsilon\dot{z}_i = g(x_i, z_i) \qquad i \in \mathcal{N} \tag{11}$$

where

$$f : \mathcal{R}^n \times \mathcal{R}^m \times \underbrace{\mathcal{R}^n \times \cdots \times \mathcal{R}^n}_{(r-1)\text{ times}} \to \mathcal{R}^n$$

$g : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}^m$. Let $f(0, \cdots, 0) = 0$, $g(0, 0) = 0$, and let $z_i = h(x_i, \cdots, x_{i-r+1})$ be an isolated root of $0 = g(x_i, z_i)$. Let $y_i = z_i - h(x_i)$, $h(0) = 0$, and $f$, $g$, and $h$ be sufficiently smooth Lipschitz functions.

*Theorem 3 (Robustness of Exponentially Stable Interconnected Systems to Singular Perturbations):* If the following conditions are satisfied:

1) Let there exist a Lyapunov function, $V(x_i)$, such that

$$\alpha_l\|x_i\|^2 \leq V(x_i) \leq \alpha_h\|x_i\|^2$$
$$\frac{\partial V}{\partial x_i}f(x_i, h(x_i), x_{i-1}, \cdots, x_{i-r+1}) \leq -\alpha_1\|x_i\|^2$$
$$+ \sum_{j=2}^{r}\alpha_{1j}\|x_{i-j+1}\|^2$$

with $\alpha_{1j} > 0$ and $\alpha_1 > \frac{\alpha_h}{\alpha_l}\sum_{j=2}^{r}\alpha_{1j}$

$$\left\|\frac{\partial V}{\partial x_i}\right\| \leq \alpha_3\|x_i\|.$$

These conditions imply the string stability of the interconnected of reduced (unperturbed) systems.

2) There exists a Lyapunov function $W(x_i, y_i)$ such that

$$\beta_l\|y_i\|^2 \leq W(x_i, y_i) \leq \beta_h\|y_i\|^2$$
$$\frac{\partial W}{\partial y_i}g(x_i, y_i + h(x_i)) \leq -\alpha_2\|y_i\|^2$$

$$\left(\frac{\partial W}{\partial x_i} - \frac{\partial W}{\partial y_i}\frac{\partial h}{\partial x_i}\right)f(x_i, y_i + h(x_i), \cdots, x_{i-r+1})$$
$$\leq \beta_2\|x_i\| \|y_i\| + \gamma\|y_i\|^2 + \sum_{j=2}^{r}\gamma_j\|x_{i-j+1}\|^2$$

with $\gamma_j > 0$. This condition implies the exponential stability of the singularly perturbed individual systems.

Then the singularly perturbed interconnected system is string stable.

*Proof:* If $\sum_{j=2}^{r} \alpha_{;1j} \neq 0$, let $k = \min\left\{\frac{\sum_{j=2}^{r} \alpha_{1j}}{\sum_{j=2}^{r} \gamma_j}, \frac{\alpha_h}{\beta_h}\right\}$.
Otherwise, let $k = \min\left\{\frac{\alpha_1}{\sum_{j=2}^{r}}, \frac{\alpha_h}{\beta_h}\right\}$. Define $\nu(x_i, y_i) = \frac{1}{2}(V(x_i) + kW(x_i, y_i))$. Using shorthand notation $\nu_i$ for $\nu(x_i, y_i)$, $V_i$ for $V(x_i)$, and $W_i$ for $W(x_i, y_i)$, there exists a $\beta_1$ such that

$$\frac{\partial V_i}{\partial x_i}[f(x_i, y_i + h(x_i), \cdots, x_{i-r+1})$$
$$- f(x_i, h(x_i), \cdots, x_{i-r+1})] \leq \beta_1 \|x_i\| \|y_i\|$$

$$\frac{\alpha_l \|x_i\|^2 + k\beta_l \|y_i\|^2}{2} \leq \nu_i \leq \frac{\alpha_h \|x_i\|^2 + k\beta_h \|y_i\|^2}{2}$$

$$\dot{\nu}_i = \frac{1}{2}[\dot{V}_i + k\dot{W}_i]$$
$$= \frac{1}{2}\left[-\alpha_1 \|x_i\|^2 + \beta_1 \|x_i\| \|y_i\| + \sum_{j=2}^{r} \alpha_{1j} \|x_{i-j+1}\|^2\right]$$
$$+ \frac{k}{2}\left[-\frac{\alpha_2}{\epsilon}\|y_i\|^2 + \beta_2 \|x_i\| \|y_i\| + \gamma \|y_i\|^2\right.$$
$$\left. + \sum_{j=2}^{r} \gamma_j \|x_{i-j+1}\|^2\right]$$
$$\leq -\lambda(\epsilon)(\|x_i\|^2 + \|y_i\|^2) + \sum_{j=2}^{r} \frac{\alpha_{1j} + k\gamma_j}{2}\|x_{i-j+1}\|^2$$

where the equation is shown at the bottom of the page. Since $\lambda(\epsilon)$ is a continuous function of $\epsilon$, define

$$F(\epsilon) = \frac{2\lambda(\epsilon)}{\alpha_h} - \sum_{j=2}^{r}(\alpha_{1j} + k\gamma_j).$$

Since $k\sum_{j=2}^{r} \gamma_j < \sum_{j=2}^{r} \alpha_{1j}$ from Assumption 1, it follows that $F(0) > 0$ and $F\left(\frac{4\alpha_1\alpha_2 k}{4\alpha_1\gamma k+(\beta_1+\beta_2 k)^2}\right) < 0$. By the intermediate value theorem, there exists $\epsilon_d$ such that $0 < \epsilon_d < \frac{4\alpha_1\sigma_2 k}{4\alpha_1\gamma k+(\beta_1+\beta_2 k)^2}$ and $\forall 0 < \epsilon < \epsilon_d$, $F(\epsilon) > 0$. Therefore

$$\dot{\nu}_i \leq -\lambda(\epsilon)(\|x_i\|^2 + \|y_i\|^2) + \sum_{j=2}^{r}(\alpha_{1j} + k\gamma_j)\frac{\|x_{i-j+1}\|^2}{2}$$
$$\leq -\frac{2\lambda(\epsilon)}{\alpha_h}\nu_i + \sum_{j=2}^{r}(\alpha_{1j} + k\gamma_j)\nu_{i-j+1}.$$

By an argument similar to that in Theorem 1, there exists a constant $K > 0$, such that $\|\nu_i(\cdot)\|_\infty \leq K\|\nu_i(0)\|_\infty$. This proves that the interconnection of singularly perturbed systems is string stable $\forall 0 < \epsilon < \epsilon_d$.

It also follows, by an argument similar to that in Theorem 1, that $\nu_i \to 0$ exponentially.

The above theorem justifies the use of control based on the reduced (unperturbed) system model.

## IV. ADAPTIVE CONTROL OF INTERCONNECTED SYSTEMS

Consider the following open-loop interconnected system:

$$\dot{\xi}_i = f_o(\xi_i, \xi_{i-1}, \cdots, \xi_{i-r+1}) + g(\xi_i)u_i$$

where $\xi_i \in \mathcal{R}^{p+q+1}$ and where $p$ and $q$ are positive integers. As assumed earlier, $\xi_j \equiv 0$ for all $j \leq 0$. $f_o$, $g$ are smooth vector fields, $u_i \in \mathcal{R}$ and $i \in \mathcal{N}$. The output of the $i$th subsystem is $h_i = h(\xi_i)$ with $h_i \in \mathcal{R}$. The objective is to find a control such that the states of the closed-loop interconnected system are always bounded and go to the origin, $\xi_i = 0$, $i \in \mathcal{N}$, asymptotically.

The following assumptions are used for obtaining the control effort and analyzing the closed-loop behavior of the interconnected system:

- There exists a global diffeomorphism $z_i = \phi_1(\xi_i)$, $y_i = \phi_2(\xi_i)$ with $z_i \in \mathcal{R}^{p+1}$, $y_i \in \mathcal{R}^q$, and

$$\dot{z}_i^{(1)} = z_i^{(2)}$$
$$\dot{z}_i^{(2)} = z_i^{(3)}$$
$$\vdots$$
$$\dot{z}_i^{(p)} = z_i^{(p+1)}$$
$$\dot{z}_i^{(p+1)} = \theta_f^T W_f(\xi_i, \xi_{i-1}, \cdots, \xi_{i-r+1}) + \theta_g^T W_g(\xi_i)u_i$$
$$\dot{y}_i = \eta(z_i, y_i)$$

where $z_i^{(j)}$ is the $j$th component of the vector $z_i$ and $z_i^{(1)} = h_i$. The above condition implies the adaptive linearizability of the open-loop system with a strict relative degree equal to $p + 1$. For details on adaptive linearizability, see Sastry and Isidori [22]. The vector fields, $f_o$, $g$ are implicitly assumed to be linearly parameterizable in the constant parameters and will, henceforth, be represented by $\hat{\theta}_f$ and $\hat{\theta}_g$, respectively. Similarly, the parameter estimation errors are given by $\theta_f$, and $\theta_g$. $y_i$ represents the state that will be rendered unobservable by an input–output linearizing control. In other words, the dynamics of $y_i$ represent the internal dynamics of the $i$th system.

- (The origin of) $\dot{y}_i = \eta(0, y_i)$ is globally exponentially stable. This assumption states that the zero dynamics of every system in the open-loop interconnected system is exponentially stable. This assumption is required to establish that $y_i$ is bounded uniformly in $i$ when $z_i$ is uniformly bounded in $i$. A more general form of internal dynamics that arises in such interconnected systems is of the following form:

$$\dot{y}_i = \eta(z_i, y_i, z_{i-1}, y_{i-1}, \cdots, z_{i-r+1}, y_{i-r+1}).$$

To analyze the closed-loop interconnected system with such an internal dynamics, additional weak coupling conditions similar to those in Theorem 1 (on the magnitude

$$\lambda(\epsilon) = \frac{4\alpha_1 k(\alpha_2 - \epsilon\gamma) - \epsilon(\beta_1 + \beta_2 k)^2}{4(\epsilon\alpha_1 + k(\alpha_2 - \epsilon\gamma) + \sqrt{(\epsilon\alpha_1 - k(\alpha_2 - \epsilon\gamma))^2 + \epsilon^2(\beta_1 + \beta_2 k)^2})}$$

of the Lipschitz constants associated with $y$ arguments of $\eta$) have to be imposed to conclude that $y_i$ is uniformly bounded in $i$ if $z_i$ are uniformly bounded in $i$. To keep the analysis simple, we will, however, not use this general form of internal dynamics.

- Every system avails the information of its state and the information of the states of "$r$" systems preceding it. This assumption is necessary to generate a feedback linearizing law which guarantees that the states of all the systems are bounded uniformly in $i$. The above assumption enables us to define $S_i$ such that $S_i = 0$ describes the desired closed-loop (string stable) dynamics. For example, one could define

$$S_i = z_i^{(p)} + \delta_1 z_i^{(p-1)} + \cdots + \delta_p z_i^{(1)} - \delta_{p+1} z_{i-1}^{(1)}$$

where $s^p + \delta_1 s^{p-1} + \cdots + \delta_p$ is a Hurwitz polynomial with real roots. On the surface, $S_i = 0$, $\|z_i\|_\infty < \|z_{i-1}\|_\infty$ if $\delta_{p+1}$ is sufficiently small. In matrix form, $S_i$ can be defined compactly as

$$\dot{x}_i = f_d(x_i, \cdots, x_{i-r+1}) + b_\lambda S_i$$

where $b_\lambda = [0, \cdots, 0, 1]^T$ and $x_i = [z_i^{(1)}, \cdots, z_i^{(p)}]^T$. $f_d$ is a smooth vector field and it satisfies the weak coupling conditions described in Theorem 1 so that the dynamics on the surface $S_i = 0$ is string stable. Algebraically, $S_i$ should be understood as

$$S_i = z_i^{(p)} + \psi_d(x_i, x_{i-1}, \cdots, x_{i-r+1}).$$

Here $\psi_d$ is a smooth scalar function.

The control input $u_i$ should be chosen to drive $S_i$ to the surface $S_i = 0$. To obtain the control effort, differentiate $S_i$

$$\begin{aligned}
\dot{S}_i &= z_i^{(p+1)} + \dot{\psi}_d(x_i, \cdots, x_{i-r+1}) \\
&= \theta_f^T W_f(\xi_i, \cdots, \xi_{i-r+1}) + \theta_g^T W_g(\xi_i) u_i \\
&\quad + \dot{\psi}_d(x_i, \cdots, x_{i-r+1}).
\end{aligned}$$

Choose $u_i$ such that

$$\begin{aligned}
&\hat{\theta}_f^T W_f(\xi_i, \cdots, \xi_{i-r+1}) + \hat{\theta}_g^T W_g(\xi_i) u_i \\
&\quad + \dot{\psi}_d(x_i, \cdots, x_{i-r+1}) = -\lambda S_i.
\end{aligned}$$

Obtaining control effort requires inversion of $\hat{\theta}_g W_g$ which may be singular. If it is known that $|\theta_g W_g| > C$ where $C$ is a generic positive constant, projection algorithms could be employed to counter this problem.

As seen earlier, the closed-loop dynamics of any adaptively linearizable nonlinear systems with a coupling (interconnecting) control law can be cast in the following form:

$$\begin{aligned}
\dot{x}_i &= f_d(x_i, x_{i-1}, \cdots, x_{i-r+1}) + b_\lambda S_i \\
\dot{S}_i &= -\lambda S_i + \tilde{\theta}_i^T W(x_i, y_i, S_i, x_{i-1}, \cdots, x_{i-r+1}) \\
\dot{y}_i &= \phi(x_i, y_i, S_i)
\end{aligned} \qquad (12)$$

where $b_\lambda = [0 \cdots 0 \ 1]^T$, $\tilde{\theta}_i = \hat{\theta}_i - \theta_i$ where $\hat{\theta}_i$ is the estimate of the parameter, and $\theta_i$ is the actual (constant) value of the parameter. From the first equation, $S_i = 0$ describes the desired closed-loop dynamics. The second equation describes the dynamics of $S_i$, and the third equation indicates the

behavior of the internal dynamics associated with this system. Any adaptively linearizable nonlinear system with a coupling (interconnecting) control law yields this form of equations. To analyze the effect of parameter adaptation, we assume the following:

1) There exists a Lyapunov function $V(x_i)$ (for convenience, $V_i$), such that

$$\begin{aligned}
&\alpha_l \|x_i\|^2 \leq V_i \leq \alpha_i \|x_i\|^2 \\
&\frac{\partial V_i}{\partial x_i} f_d(x_i, x_{i-1}, \cdots, x_{i-r+1}) \\
&\quad \leq -l_1 \|x_i\|^2 + \sum_{j=2}^{r} l_j \|x_{i-j+1}\|^2 \\
&\left\| \frac{\partial V_i}{\partial x_i} \right\| \leq \alpha_1 \|x_i\|
\end{aligned}$$

with $l_1 > \frac{\alpha_h}{\alpha_l} \sum_{j=2}^{r} l_j$.

2) There exists a Lyapunov function $W_z(y_i)$ (for convenience, $W_i$), such that

$$\begin{aligned}
&\beta_l \|y_i\|^2 \leq W_i \leq \beta_h \|y_i\|^2 \\
&\frac{\partial W_i}{\partial y_i} \phi(x_i, S_i, y_i) \leq -\alpha_2 \|y_i\|^2 + \alpha_3 \|y_i\| \, |S_i| \\
&\qquad\qquad\qquad\qquad + \alpha_4 \|y_i\| \, \|x_i\| \\
&\left\| \frac{\partial W_i}{\partial y_i} \right\| \leq \alpha_5 \|y_i\|.
\end{aligned}$$

We assume the exponentially stable behavior of the zero dynamics. Assumptions 1 and 2 enable the string stability of the interconnected system in the absence of parameter mismatch.

3) $W(x_i, y_i, S_i, x_{i-1}, \cdots, x_{i-r+1})$ is bounded for all its bounded arguments.

*Theorem 4 (Effectiveness of Parameter Adaptation for Interconnected Systems):* Under the above mentioned conditions, the following parameter adaptation law:

$$\dot{\hat{\theta}}_i = -\Gamma W(x_i, \cdots, x_{i-r+1}) S_i, \qquad \Gamma > 0$$

guarantees that for all bounded $\|x_i(0)\|_\infty$, $\|S_i(0)\|_\infty$, $\|\tilde{\theta}_i(0)\|_\infty$

- $\sup_i \|x_i(\cdot)\|_\infty$, $\sup_i \|S_i(\cdot)\|_\infty$, $\sup_i \|\tilde{\theta}_i(\cdot)\|_\infty$ are bounded;
- $x_i(t)$, $S_i(t) \to 0$ asymptotically for all $i$.

*Proof:* Let $V_{ai} = S_i^2 + \tilde{\theta}_i^T \Gamma^{-1} \tilde{\theta}_i$. Using the adaptation law

$$\dot{V}_{ai} = -2\lambda S_i^2$$

$$\Rightarrow V_{ai}(t) \leq V_{ai}(0) \leq S_i^2(0) + \frac{\|\tilde{\theta}_i(0)\|^2}{\lambda_{\min}(\Gamma)}$$

$$\sup_i \|S_i(\cdot)\|_\infty \leq \sqrt{\|S_i(0)\|_\infty^2 + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}.$$

Similarly

$$\sup_i \|\tilde{\theta}_i(\cdot)\|_\infty \leq \sqrt{\lambda_{\max}(\Gamma)} \sqrt{\|S_i(0)\|_\infty^2 + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}$$

and

$$\sup_i \int_0^\infty S_i^2 \, dt = \sup_i \|S_i(\cdot)\|_2^2 \leq \frac{V_{ai}(0)}{2\lambda}$$

$$\leq \frac{S_i^2(0) + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}{2\lambda}.$$

Calculating $\dot{V}_i$ along the trajectories of $x_i$

$$\dot{V}_i = \frac{\partial V_i}{\partial x_i}[f_d(x_i, \cdots, x_{i-r+1}) + b_\lambda S_i]$$

$$\leq -l_1\|x_i\|^2 + \sum_{j=2}^r l_j\|x_{i-j+1}\|^2 + \alpha_1\|x_i\|\,|S_i|.$$

Since $\sup_i \|S_i\|_\infty \leq K$ where $K := \sqrt{\|S_i(0)\|_\infty^2 + \frac{\|\tilde{\theta}_i(0)\|_\infty^2}{\lambda_{\min}(\Gamma)}}$

$$\dot{V}_i \leq -l_1\|x_i\|^2 + \sum_{j=2}^r l_j\|x_{i-j+1}\|^2 + \alpha_1 K\|x_i\|.$$

Define $e_i = \sqrt{V_i}$. Then

$$\dot{e}_i \leq -\frac{l_1}{2\alpha_h}e_i + \sum_2^r \frac{l_j}{2\alpha_l}e_{i-j+1} + \frac{\alpha_1}{2\alpha_l}|S_i|$$

$$\|e_i\|_p \leq \frac{\alpha_h}{l_1}\sum_2^r \frac{l_j}{\alpha_l}\|e_{i-j+1}\|_p + \frac{\alpha_h \alpha_1}{l_1\sqrt{\alpha_l}}\|S_i\|_p$$

where $p = 2, \infty$. Since $l_1 > \frac{\alpha_h}{\alpha_l}\sum_2^r l_j\|e_i\|_p \leq M\|S_i\|_p$ where $M > 0$ is a constant. Since $\sup_i \{\|S_i\|_\infty, \|S_i\|_2\} < \max \left\{K, \sqrt{\frac{V_{ai}(0)}{2\lambda}}\right\} < \infty$, it follows that $\sup_i \{\|e_i\|_\infty, \|e_i\|_2\} < K_1$ for some positive $K_1$. This implies that $\sup_i\{\|x_i\|_\infty, \|x_i\|_2\} < \frac{K_1}{\sqrt{\alpha_l}}$. By Assumption 2 (that the zero dynamics of every individual system is minimum phase), $\sup_i \|y_i(\cdot)\|_\infty$ exists. By Assumption 3, $W(x_i, S_i, y_i, x_{i-1}, \cdots, x_{i-r+1})$ is bounded. Therefore, $S_i \in L_\infty$. Consequently, by Barbalat's lemma, $S_i \to 0$.

Observe that $\sup_i\|\dot{e}_i\|_\infty$ is bounded, since

$$\dot{e}_i \leq -\frac{l_1}{2\alpha_h}e_i + \sum_2^r \frac{l_j}{2\alpha_l}e_{i-j+1} + \frac{\alpha_1}{2\alpha_l}|S_i|.$$

Since $\sup_i \{\|e_i\|_\infty, \|e_i\|_2\}$ are bounded, by Barbalat's lemma, $e_i \to 0$. Therefore, $V_i, x_i \to 0$.

*Remarks:*

1) In Assumption 1, $S_i \equiv 0$ yields the desired "string stable" dynamics.
2) Designing decentralized adaptive controllers for interconnected systems can be done in two steps:
   a) Identify the desired closed-loop (string stable) dynamics. Design a controller to achieve the desired closed-loop dynamics in the absence of parametric uncertainty.
   b) Use a gradient adaptation law to update the parameters.

3) The dynamics of $S_i$ are usually given by

$$\dot{S}_i = -\lambda \operatorname{sign}(S_i) + \tilde{\theta}_i W(x_i, y_i, S_i, x_{i-1}, \cdots, x_{i-r+1})$$

then the following adaptation law should be used:

$$\dot{\theta}_i = -\Gamma W(x_i, \cdots, x_{i-r+1}) \operatorname{sign}(S_i), \qquad \Gamma > 0$$

to conclude that $\sup_i \|x_i\|_\infty$, $\sup_i\|S_i\|_\infty$, $\sup_i\|\tilde{\theta}\|_\infty$ are bounded and that $x_i(t)$, $S_i(t) \to 0$ asymptotically for all $i$.

The proof of the above remark is similar to the proof of Theorem 4.

## V. EXAMPLE: VEHICLE-FOLLOWING SYSTEMS

For a good overview on vehicle-following systems, the readers are referred to [2], [3], [11], [8], and [23]. The longitudinal constant spacing vehicle-following controller designed by Hedrick [8] and used for parameter adaptation by Swaroop [28] will be considered here. A simple longitudinal vehicle dynamic model for the $i$th vehicle in the platoon is given by

$$\ddot{x}_i = \frac{u_i - c_i\dot{x}_i^2 - F_i}{M_i}$$

where $x_i$ is the position of the $i$th vehicle in the platoon with respect to an inertial frame, $u_i$ represents the propulsive/braking effort, $c_i\dot{x}_i^2$ is the aerodynamic drag force, and $F_i$ is the tire drag acting on the $i$th vehicle. The control objectives are:

- $\epsilon_i(t)$ is defined as $x_i(t) - x_{i-1}(t) = L_i$, where $L_i$ is the desired constant intervehicular spacing. $\epsilon_i(t)$ should go to zero asymptotically (exponentially) for every lead vehicle maneuver.
- String stability of the platoon should be guaranteed, i.e., given $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\epsilon_i(0)\|_\infty < \delta \Rightarrow \sup_i\|\epsilon_i\|_\infty < \epsilon$.

In designing the controller [8], [28], it is assumed that the lead vehicle velocity and acceleration and lead vehicle relative position information can be communicated to every controlled vehicle. Define

$$S_i = \dot{\epsilon}_i + q_1\epsilon_i + q_3(v_i - v_l) + q_4\left(x_i - x_l + \sum_1^i L_j\right)$$

and $u_i$ is chosen such that $\dot{S}_i + \lambda S_i = 0$ for some $\lambda > 0$. $u_i$ is given by

$$u_i = c_i\dot{x}_i^2 + F_i + \frac{1}{1 + q_3}$$
$$\cdot [\ddot{x}_{i-1} + q_3\ddot{x}_l - q_1\dot{\epsilon}_i - q_4(v_i - v_l) - \lambda S_i].$$

The spacing error dynamics are given by

$$\ddot{\epsilon}_i + \left(\frac{q_1 + q_4}{1 + q_3} + \lambda\right)\dot{\epsilon}_i + \frac{\lambda(q_1 + q_4)}{q_1 + q_3}$$
$$= \frac{1}{1 + q_3}[\ddot{\epsilon}_{i-1} + (q_1 + \lambda)\dot{\epsilon}_{i-1} + q_1\lambda\epsilon_{i-1}].$$

Let $z_i = \dot{e}_i + \lambda e_i$. The dynamics of the "$z$" interconnected system is

$$\dot{z}_i = -\frac{q_1 + q_4}{1 + q_3} z_i + \frac{1}{1 + q_3} \dot{z}_{i-1} + \frac{q_1}{1 + q_3} z_{i-1}.$$

Platoon string stability is guaranteed if the above interconnected system is string stable. The above interconnected system is in the same form as in (7). String stability is guaranteed if $q_1$, $q_3$, and $q_4$ are chosen appropriately.

The theorems developed in this paper are sufficient conditions to guarantee string stability for nonlinear systems. For linear systems, one could use input–output stability and some results of this paper to conclude about string stability. For example, Laplace transformation of the above equation yields

$$\hat{Z}_i(s) = \frac{s + q_1}{s + \frac{q_1 + q_4}{1 + q_3}} \hat{Z}_{i-1}(s).$$

If $q_1$, $q_3$, $q_4$ are chosen such that $q_1 q_3 > q_4$

$$\|z_i\|_\infty \le \frac{q_1}{q_1 + q_4} \|z_{i-1}\|_\infty + K_1 |z_i(0)| + K_2 |z_{i-1}(0)|$$

for some positive $K_1$, $K_2$. String stability and, consequently, exponential string stability follow immediately.

## VI. CONCLUSIONS

In this paper we defined string stability, asymptotic, and exponential string stability, for countably infinite interconnection of nonlinear systems. We derived sufficient conditions to guarantee string stability for a class of interconnected systems and demonstrated their robustness to small singular/structural perturbations. The interconnections considered "look ahead" lower-triangular interconnected systems) or are "banded" (finite "look ahead" and "look back"). We presented parameter adaptation law (gradient type) to regulate the states of all systems and to ensure uniform boundedness of all the states and parameter estimation errors. Finally, we have illustrated the theory developed in this paper with a controller design for vehicle-following systems.

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